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New families of exact solutions to the integrable dispersive long wave equations in $(2+1)$-dimensional spaces

Bo Tian† and Yi-Tian Gao‡†

† Department of Computer Sciences, and Institute for Science & Engineering Computations, Lanzhou University, Lanzhou 730000, People's Republic of China§

‡ CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China

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Abstract. The integrable dispersive long wave equations, especially the higher dimensional ones, are of current interest in both physics and mathematics. Obtained in this paper, via a symbolic-computation-based method, are new families of exact solutions to the $(2+1)$-dimensional integrable dispersive long wave equations. Sample solutions from those families are presented. Solitary waves are merely a special case in one family.

1. Introduction

The integrable dispersive long wave equations are interesting topics in physics and mathematics, while the application of symbolic computation to physical and mathematical sciences appears to have a bright future.

Since the 1960's, many one-dimensional versions of the dispersive long wave equations have been proposed to model the water wave propagation in certain infinitely-long channels of finite constant depth and narrow width. Those equations are found integrable, to have some soliton solutions and plenty of mathematical properties associated with the infinite-dimensional completely integrable Hamiltonian systems [1–6]. Lately, improvement has been made in the stability theory for solitary-wave solutions of model equations for long waves [7].

Recently, to cover wide channels or open seas, that system has been extended to the coupled integrable dispersive long wave equations in $(2+1)$-dimensional spaces (thereafter IDLWE), as a compatibility condition for a weak Lax pair [8],

$$u_{ty} = -\eta_{xx} - \tfrac{1}{2}(u^2)_{xy} \tag{1}$$

$$\eta_t = -(u\eta + u + u_{xy})_x \tag{2}$$

where $\eta(x, y, t)$ represents the amplitude of a surface wave, which propagates in the (x, y) plane, with its horizontal velocity as $u(x, y, t)$. A relevant Kac–Moody–Virasoro symmetry algebra has been studied [9], followed by a set of generalized symmetries constituting an infintely-dimensional Lie algebra [10]. It is interesting to note that the IDLWE has no Painlevé property [11] even if it is integrable [8].

In this paper, we use a symbolic-computation-based method to obtain new families of exact solutions to the IDLWE, and to give examples from those families.

§ Mailing address (correspondence to Yi-Tian Gao).

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2. Leading to the format

The following essential ideas of two direct methods are important: (1) the Hirota's dependent variable transformation introduces, to begin with, a dependent variable $z(x,t)$ with a differentiator acting on its function $w[z(x,t)] = \ln z(x,t)$; (2) the Clarkson–Kruskal approach considers a general function $F\{x, t, w[z(x,t)]\}$ and tries to establish an ordinary differential equation (ODE) for $w(z)$ so as to impose conditions upon F and z. For a review, see [12–14].

Enlightened by those ideas, we consider the following generic transformations for a set of coupled partial differential equations (PDEs):

$$\begin{cases} u(x,y,t) = \Omega(\partial_t, \partial_x, \partial_y)\{w[z(x,y,t)]\} \\ \eta(x,y,t) = \Upsilon(\partial_t, \partial_x, \partial_y)\{w[z(x,y,t)]\} \end{cases} \tag{3}$$

where Ω and Υ are a couple of operators to be determined, and $z(x,y,t)$ is a dependent variable also to be determined. Then, we try to substitute equations (3) back into the original PDEs, in order to obtain a set of coupled ODEs for $w(z)$ and the conditions upon Ω, Υ and $z(x,y,t)$. A more powerful alternative is to determine, first and foremost, some of Ω, Υ and z, which would reduce the amount of computations quite a lot.

It is well known that the condition for the soliton-like solutions to occur is that the effects of different mechanisms that act to change wave forms, i.e. dispersion, dissipation and nonlinearity, either separately or in various combinations, are able to exactly balance out ([15–18] for a general review).

To get the expression for Ω first, we consider the leading-order conjecture that the balancing act concentrates on the terms with the highest powers of the differential coefficients of $z(x,y,t)$ (to be seen in the forthcoming analysis), which seem to dominate the aforementioned effects. For equations (1) and (2), we assume that equations (3) have a special format,

$$\begin{cases} u(x,y,t) = A\partial_x^m \partial_y^n w[z(x,y,t)] + B \\ \eta(x,y,t) = C\partial_x^j \partial_y^l w[z(x,y,t)] + D \end{cases} \tag{4}$$

where the constants A, B, C and D, as well as the integers j, l, m and n, are to be determined later.

The leading-order analysis is performed as follows. For equation (1), the (possible) highest-power terms are $z_x^{j+2} z_y^l$ and $z_x^{2m+1} z_y^{2n+1}$, which are, respectively, contributed by η_{xx} and $(u^2)_{xy}$. No term with so high a power is seen from u_{ty}. Then, the balancing act requires that those two terms have the same power, i.e. $j = 2m-1$ and $l = 2n+1$. Similarly, for equation (2), the (possible) highest-power terms are $z_x^{m+j+1} z_y^{n+l}$ and $z_x^{m+2} z_y^{n+1}$, contributed by $(u\eta)_x$ and u_{xxy}, yielding $j = 1$ and $l = 1$. No term with so high a power is seen from either η_t or u_x. Therefore, we conclude the analysis with $j = l = m = 1$ and $n = 0$, so as to get

$$\begin{cases} u(x,y,t) = A\partial_x w[z(x,y,t)] + B \\ \eta(x,y,t) = C\partial_x \partial_y w[z(x,y,t)] + D. \end{cases} \tag{5}$$

That format has been briefly mentioned in [19, 20], and extensively investigated here so as to allow us to find out the conditions imposed upon $z(x,y,t)$, along with the determination of the $w(z)$'s coupled ODEs.

3. The treatment of $w(z)$ and $z(x, y, t)$

With the aid of MATHEMATICA, equations (1) and (2) become the set of

$$\begin{aligned}
&Aw'''z_tz_yz_x + Aw''z_{yt}z_x + ABw'''z_yz_x^2 + A^2(w'')^2z_yz_x^3 + A^2w'w'''z_yz_x^3\\
&\quad+Cw^{(4)}z_yz_x^3 + Aw''z_yz_{xt} + Aw''z_tz_{xy} + 2ABw''z_xz_{xy} + 3A^2w'w''z_x^2z_{xy}\\
&\quad+3Cw'''z_x^2z_{xy} + Aw'z_{xyt} + ABw''z_yz_{xx} + 2A^2w'w''z_yz_xz_{xx}\\
&\quad+3Cw'''z_yz_xz_{xx} + A^2(w')^2z_{xy}z_{xx} + 3Cw''z_{xy}z_{xx} + ABw'z_{xxy}\\
&\quad+A^2(w')^2z_xz_{xxy} + 3Cw''z_xz_{xxy} + Cw''z_yz_{xxx} + Cw'z_{xxxy} = 0 \qquad (6)
\end{aligned}$$

$$\begin{aligned}
&Cw'''z_tz_yz_x + Cw''z_{yt}z_x + Aw''z_x^2 + ADw''z_x^2 + BCw'''z_yz_x^2 + AC(w'')^2z_yz_x^3\\
&\quad+ACw'w'''z_yz_x^3 + Aw^{(4)}z_yz_x^3 + Cw''z_yz_{xt} + Cw''z_tz_{xy}\\
&\quad+2BCw''z_xz_{xy}3ACw'w''z_x^2z_{xy} + 3Aw'''z_x^2z_{xy} + Cw'z_{xyt}\\
&\quad+Aw'z_{xx} + ADw'z_{xx} + BCw''z_yz_{xx} + 2ACw'w''z_yz_xz_{xx} + 3Aw'''z_yz_xz_{xx}\\
&\quad+AC(w')^2z_{xy}z_{xx} + 3Aw''z_{xy}z_{xx} + BCw'z_{xxy} + AC(w')^2z_xz_{xxy}\\
&\quad+3Aw''z_xz_{xxy} + Aw''z_yz_{xxx} + Aw'z_{xxxy} = 0 \qquad (7)
\end{aligned}$$

where the prime notation represents the differentiation with respect to z. We equate to zero the terms with the highest power of the differential coefficients of $z(x, y, t)$, i.e. the $z_x^3z_y$ terms, so as to get

$$A^2(w'')^2 + A^2w'w''' + Cw^{(4)} = 0 \qquad \text{and} \qquad C(w'')^2 + Cw'w''' + w^{(4)} = 0. \qquad (8)$$

This is the coupled ODE system we are looking for, of which a special solution is of the form

$$w(z) = k \cdot \ln(z) \qquad (9)$$

with the constants calculated as

$$k = \frac{2}{C} \qquad \text{and} \qquad A = \pm C. \qquad (10)$$

It is noted that if one neglects the zero and linear solutions of equations (8) then, if (8) have a solution, it is necessary that $A = \pm C$.

Having seen the expression for $w(z)$, we investigate $z(x, y, t)$. The equation splitting [21] is applied to the remainders of equations (6) and (7), with their simplest parts, or the w' terms, vanishing, i.e.

$$Az_{xyt} + ABz_{xxy} + Cz_{xxxy} = 0 \qquad (11)$$

$$Cz_{xyt} + Az_{xx} + ADz_{xx} + BCz_{xxy} + Az_{xxxy} = 0. \qquad (12)$$

With the choices of

$$B = 0 \qquad D = -1 \qquad \text{and} \qquad A = C \qquad (13)$$

one can see that the simple trial solution for equations (11) and (12),

$$z(x, y, t) = 1 + \exp(\alpha \cdot x + \beta \cdot y + \gamma \cdot t + \delta) \qquad (14)$$

would lead to nothing but solitary waves, where α, β, γ and δ are constants.

Nevertheless, this clue inspires one to proceed further. More sophisticated than solitary waves, the following x-linear form under constraints (13),

$$z(x, y, t) = \Sigma(y, t) + \exp[\Theta(y, t) \cdot x + \Psi(y, t)] \qquad (15)$$

is now imposed *a priori* for the search of particular solutions to system (1)–(2), where $\Sigma(y, t)$, $\Theta(y, t)$ and $\Psi(y, t)$ are differentiable functions of y and t only. In other words, we proceed directly by assuming a solution to equations (1) and (2) of the form given by equations (5), (9), (10) and (13) with z given by equation (15). However, equations (11) and (12) will be ignored, as they become unuseful in the forthcoming analysis.

4. A formalism of solutions for the IDLWE

After the substitution of equations (5), (9), (10), (13) and (15) with symbolic computation, we find that equations (1) and (2) give rise to the same large equation as follows:

$$
\begin{aligned}
&3e^{\Theta x+\Psi}\Theta^2\Sigma\Theta_y - e^{\Theta x+\Psi}x\Theta^3\Sigma\Theta_y + 3\Theta^2\Sigma^2\Theta_y + x\Theta^3\Sigma^2\Theta_y + 2e^{\Theta x+\Psi}x\Sigma\Theta_t\Theta_y\\
&\quad -e^{\Theta x+\Psi}x^2\Theta\Sigma\Theta_t\Theta_y + 2x\Sigma^2\Theta_t\Theta_y + x^2\Theta\Sigma^2\Theta_t\Theta_y + e^{\Theta x+\Psi}\Sigma\Psi_t\Theta_y\\
&\quad -e^{\Theta x+\Psi}x\Theta\Sigma\Psi_t\Theta_y + \Sigma^2\Psi_t\Theta_y + x\Theta\Sigma^2\Psi_t\Theta_y - e^{\Theta x+\Psi}\Sigma_t\Theta_y\\
&\quad +e^{\Theta x+\Psi}x\Theta\Sigma_t\Theta_y - \Sigma\Sigma_t\Theta_y - x\Theta\Sigma\Sigma_t\Theta_y - e^{\Theta x+\Psi}\Theta^3\Sigma\Psi_y + \Theta^3\Sigma^2\Psi_y\\
&\quad +e^{\Theta x+\Psi}\Sigma\Theta_t\Psi_y - e^{\Theta x+\Psi}x\Theta\Sigma\Theta_t\Psi_y + \Sigma^2\Theta_t\Psi_y + x\Theta\Sigma^2\Theta_t\Psi_y\\
&\quad -e^{\Theta x+\Psi}\Theta\Sigma\Psi_t\Psi_y + \Theta\Sigma^2\Psi_t\Psi_y + e^{\Theta x+\Psi}\Theta\Sigma_t\Psi_y - \Theta\Sigma\Sigma_t\Psi_y\\
&\quad +e^{\Theta x+\Psi}\Theta^3\Sigma_y - \Theta^3\Sigma\Sigma_y - e^{\Theta x+\Psi}\Theta_t\Sigma_y + e^{\Theta x+\Psi}x\Theta\Theta_t\Sigma_y - \Sigma\Theta_t\Sigma_y\\
&\quad -x\Theta\Sigma\Theta_t\Sigma_y + e^{\Theta x+\Psi}\Theta\Psi_t\Sigma_y - \Theta\Sigma\Psi_t\Sigma_y + 2\Theta\Sigma_t\Sigma_y + (e^{\Theta x+\Psi})^2\Theta_{yt}\\
&\quad +2e^{\Theta x+\Psi}\Sigma\Theta_{yt} + \Sigma^2\Theta_{yt} + e^{\Theta x+\Psi}x\Sigma\Theta_{yt} + x\Theta\Sigma^2\Theta_{yt} + e^{\Theta x+\Psi}\Theta\Sigma\Psi_{yt}\\
&\quad +\Theta\Sigma^2\Psi_{yt} - e^{\Theta x+\Psi}\Theta\Sigma_{yt} - \Theta\Sigma\Sigma_{yt} = 0.
\end{aligned}
\tag{16}
$$

This equation is going to be satisfied if the terms with $e^{\Theta x+\Psi}x^2$, $e^{\Theta x+\Psi}x$, $(e^{\Theta x+\Psi})^2$, $e^{\Theta x+\Psi}$, x^2, x and x^0 are assumed to vanish separately. Correspondingly, equation (16) becomes a set of constraints, after some algebraic manipulations and reductions:

$$
\begin{cases}
\Theta_t\Theta_y = 0 \qquad \text{(so that } \Theta_{yt} = 0 \text{ is also satisfied)}\\
-\Theta^2\Sigma\Theta_y - \Sigma\Psi_t\Theta_y + \Sigma_t\Theta_y - \Sigma\Theta_t\Psi_y + \Theta_t\Sigma_y = 0\\
-\Theta^2\Sigma^2\Psi_y - \Sigma^2\Psi_t\Psi_y + \Sigma\Sigma_t\Psi_y + \Theta^2\Sigma\Sigma_y + \Sigma\Psi_t\Sigma_y - \Sigma_t\Sigma_y = 0\\
2\Theta\Sigma^2\Theta_y + \Sigma_t\Sigma_y + \Sigma^2\Psi_{yt} - \Sigma\Sigma_{yt} = 0.
\end{cases}
\tag{17}
$$

To this stage, we are able to present the formalism of new solutions for equations (1) and (2) as follows:

$$u(x, y, t) = A\partial_x w[z(x, y, t)] = \frac{2\cdot\Theta(y, t)\cdot e^{\Theta(y,t)\cdot x+\Psi(y,t)}}{e^{\Theta(y,t)\cdot x+\Psi(y,t)} + \Sigma(y, t)} \tag{18}$$

$$\eta(x, y, t) = A\partial_x\partial_y w[z(x, y, t)] - 1 = \frac{2\cdot\Xi(x, y, t)\cdot e^{\Theta(y,t)\cdot x+\Psi(y,t)}}{[e^{\Theta(y,t)\cdot x+\Psi(y,t)} + \Sigma(y, t)]^2} - 1 \tag{19}$$

where the differentiable functions $\Sigma(y, t)$, $\Theta(y, t)$, $\Psi(y, t)$ and their derivatives are mutually linked through constraints (17), while the last function is defined as

$$
\begin{aligned}
\Xi(x, y, t) = {}& e^{\Theta(y,t)\cdot x+\Psi(y,t)}\Theta_y(y, t) + \Sigma(y, t)\Theta_y(y, t) + x\cdot\Theta(y, t)\Sigma(y, t)\Theta_y(y, t)\\
&+\Theta(y, t)\Sigma(y, t)\Psi_y(y, t) - \Theta(y, t)\Sigma_y(y, t).
\end{aligned}
\tag{20}
$$

The case with $\Sigma(y, t) = 0$ is too simple to be of interest. We will concentrate on the general case of $\Sigma(y, t) \neq 0$, when the formalism turns out to be

$$u(x, y, t) = \Theta(y, t)\left\{\tanh\left[\frac{\Theta(y, t)\cdot x + \Psi(y, t) - \ln\Sigma(y, t)}{2}\right] + 1\right\} \tag{21}$$

$$\eta(x, y, t) = \frac{\Xi(x, y, t)}{2\cdot\Sigma(y, t)}\,\mathrm{sech}^2\left[\frac{\Theta(y, t)\cdot x + \Psi(y, t) - \ln\Sigma(y, t)}{2}\right] - 1. \tag{22}$$

The physical interest of such solutions lies in the fact that they describe certain *soliton-like* surface waves, the crests of which acquire shape in the (x, y) plane. The actual form of the amplitude depends on the choices of $\Xi(x, y, t)$ and $\Sigma(y, t)$, while its horizontal velocity on $\Theta(y, t)$. Detailed discussions now follow.

5. Families of solutions to the IDLWE

In order to make $\Theta_t\Theta_y = 0$, expressions (21) and (22), along with constraints (17) and equation (20), lead to three families of exact solutions to equations (1) and (2). They are separately characterized by $\Theta =$ constant; $\Theta_t = 0$ but $\Theta_y \neq 0$; and $\Theta_y = 0$ with $\Theta_t \neq 0$.

Family I: $\Theta(y, t) = \theta = constant \neq 0$

For this family, constraints (17) reduce to

$$\begin{cases} -\theta^2\Sigma^2\Psi_y - \Sigma^2\Psi_t\Psi_y + \Sigma\Sigma_t\Psi_y + \theta^2\Sigma\Sigma_y + \Sigma\Psi_t\Sigma_y - \Sigma_t\Sigma_y = 0 \\ \Sigma_t\Sigma_y + \Sigma^2\Psi_{yt} - \Sigma\Sigma_{yt} = 0 \end{cases} \tag{23}$$

and the exact solutions are written as

$$u^{(I)}(x, y, t) = \theta \cdot \left\{\tanh\left[\frac{\theta \cdot x + \Psi(y, t) - \ln\Sigma(y, t)}{2}\right] + 1\right\} \tag{24}$$

$$\eta^{(I)}(x, y, t) = \frac{\theta}{2} \cdot \left[\Psi_y(y, t) - \frac{\Sigma_y(y, t)}{\Sigma(y, t)}\right] \cdot \text{sech}^2\left[\frac{\theta \cdot x + \Psi(y, t) - \ln\Sigma(y, t)}{2}\right] - 1 \tag{25}$$

where the differentiable functions $\Sigma(y, t)$, $\Psi(y, t)$ and their derivatives satisfy constraints (23).

Sample 1.1. We have a choice of

$$\Sigma(y, t) = y \tag{26}$$

by which the second of equations (23) becomes

$$\Psi_{yt} = 0 \tag{27}$$

or

$$\Psi(y, t) = \Gamma(y) + \Phi(t) \tag{28}$$

where $\Gamma(y)$ and $\Phi(t)$ are arbitrary, differentiable functions. Substitute expression (28) back into the first of equations (23), and we find that

$$\Psi(y, t) = \Gamma(y) - \theta^2 t. \tag{29}$$

As a result, the first sample of family I turns out to be

$$u^{(I)}(x, y, t) = \theta \cdot \left\{\tanh\left[\frac{\theta \cdot x + \Gamma(y) - \theta^2 t - \ln y}{2}\right] + 1\right\} \tag{30}$$

$$\eta^{(I)}(x, y, t) = \frac{\theta}{2y} \cdot [y\Gamma_y(y) - 1] \cdot \text{sech}^2\left[\frac{\theta \cdot x + \Gamma(y) - \theta^2 t - \ln y}{2}\right] - 1. \tag{31}$$

Sample 1.2. Solitary waves. Let us assume that

$$\Psi(y,t) = ay + bt + c \qquad \text{and} \qquad \Sigma(y,t) = 1 \tag{32}$$

where a, b and c are constants. Some of them are arbitrary. Substitute equations (32) back into equations (23), and we find that

$$b = -\theta^2. \tag{33}$$

Therefore,

$$u^{(I)}(x,y,t) = \theta \cdot \left[\tanh\left(\frac{\theta x + ay - \theta^2 t + c}{2}\right) + 1\right] \tag{34}$$

$$\eta^{(I)}(x,y,t) = \frac{a\theta}{2} \cdot \mathrm{sech}^2\left(\frac{\theta x + ay - \theta^2 t + c}{2}\right) - 1. \tag{35}$$

Thus, solitary waves are nothing but a special case of family I.

Family II. $\Theta = \Theta(y)$ *only so as to make* $\Theta_y \neq 0$ *while* $\Theta_t = 0$

This time, constraints (17) reduce to

$$\begin{cases} -\Theta^2\Sigma - \Sigma\Psi_t + \Sigma_t = 0 \\ \Sigma^2(\Theta^2)_y + \Sigma_t\Sigma_y + \Sigma^2\Psi_{yt} - \Sigma\Sigma_{yt} = 0 \end{cases} \tag{36}$$

and we get

$$u^{(II)}(x,y,t) = \Theta(y) \cdot \left\{\tanh\left[\frac{\Theta(y)\cdot x + \Psi(y,t) - \ln\Sigma(y,t)}{2}\right] + 1\right\} \tag{37}$$

$$\begin{aligned} \eta^{(II)}(x,y,t) = {} & \frac{1}{2\cdot\Sigma(y,t)}\,\mathrm{sech}^2\left[\frac{\Theta(y)\cdot x + \Psi(y,t) - \ln\Sigma(y,t)}{2}\right] \\ & \cdot[\mathrm{e}^{\Theta(y)\cdot x + \Psi(y,t)}\Theta_y(y) + \Sigma(y,t)\Theta_y(y) + x\cdot\Theta(y)\Sigma(y,t)\Theta_y(y) \\ & + \Theta(y)\Sigma(y,t)\Psi_y(y,t) - \Theta(y)\Sigma_y(y,t)] - 1 \end{aligned} \tag{38}$$

where the differentiable functions $\Theta(y)$, $\Sigma(y,t)$, $\Psi(y,t)$ and their derivatives satisfy constraints (36).

Sample 2. We now select

$$\Sigma(y,t) = y \cdot t \tag{39}$$

and simplify constraints (36) to

$$\begin{cases} -\Theta^2 t - \Psi_t \cdot t + 1 = 0 \\ (\Theta^2)_y + \Psi_{yt} = 0. \end{cases} \tag{40}$$

It is noted that the integration of the second of equations (40) over y leads to the first of equations (40). The second time of integration, over t, further leads to the expression

$$\Psi(y,t) = \ln t - \Theta^2(y)\cdot t + \Phi(y) \tag{41}$$

so that

$$u^{(II)}(x,y,t) = \Theta(y)\cdot\left\{\tanh\left[\frac{\Theta(y)\cdot x - \Theta^2(y)\cdot t + \Phi(y) - \ln y}{2}\right] + 1\right\} \tag{42}$$

$$\begin{aligned}\eta^{(II)}(x,y,t) = {} & \frac{1}{2y}\,\mathrm{sech}^2\left[\frac{\Theta(y)\cdot x - \Theta^2(y)\cdot t + \Phi(y) - \ln y}{2}\right] \\ & \cdot[\mathrm{e}^{\Theta(y)\cdot x - \Theta^2(y)\cdot t + \Phi(y)}\Theta_y(y) - \Theta(y) + y\Theta_y(y) + xy\Theta(y)\Theta_y(y) \\ & -2ty\Theta^2(y)\Theta_y(y) + y\Theta(y)\Phi_y(y)] - 1\end{aligned} \tag{43}$$

where $\Phi(y)$ and $\Theta(y)$ are arbitrary, differentiable functions of y.

Family III. $\Theta = \Theta(t)$ *only so as to make*

$\Theta_y = 0$ *while* $\Theta_t \neq 0$

Constraints (17) reduce to

$$\begin{cases} -\Sigma\Psi_y + \Sigma_y = 0 \\ \Sigma_t\Sigma_y + \Sigma^2\Psi_{yt} - \Sigma\Sigma_{yt} = 0. \end{cases} \tag{44}$$

Investigation shows that the first derivative of the first of equations (44) with respect to t gives rise to the second of equations (44), while its first integration over y yields

$$\Sigma(y,t) = \Phi(t)\mathrm{e}^{\Psi(y,t)} \tag{45}$$

which leads to the results

$$u^{(III)}(x,y,t) = \Theta(t)\cdot\left\{\tanh\left[\frac{\Theta(t)\cdot x - \ln\Phi(t)}{2}\right] + 1\right\} \tag{46}$$

$$\eta^{(III)}(x,y,t) = -1 \tag{47}$$

where $\Theta(t)$ and $\Phi(t)$ are arbitrary, differentiable functions. In this family, $u^{(III)} = u(x,t)$ is independent of y, and $\eta^{(III)}$ is only a constant.

Note. All of our results have been verified with respect to the original IDLWE, i.e. equations (1) and (2), by virtue of MATHEMATICA. See the appendix.

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Appendix

All the results presented in this paper as claimed, have been verified by the use of MATHEMATICA. The following is an example.

Sample 1.1, for instance, can be straightforwardly verified with the MATHEMATICA program below, in which *Eqn1Verify* and *Eqn2Verify* of the command lines In[1] and In[3] are the user-defined verification functions for equations (1) and (2), while u_{1a} and η_{1a} of the In[4] and In[5] lines represent equations (30) and (31), respectively. The results Out[6] and Out[8] thus indicate that u_{1a} and η_{1a} are indeed a set of particular solutions for equations (1) and (2).

```
In[1]:=
Eqn1Verify[u_,eta_]:=Simplify[D[u,{t,1},{y,1}]\
+D[eta,{x,2}]+(1/2)*D[(u^2),{x,1},{y,1}]]

In[2]:=
www[u_,eta_]:=Simplify[u*eta+u+D[u,{x,1},{y,1}]]

In[3]:=
Eqn2Verify[u_,eta_]:=Simplify[D[eta,t]+D[www[u,eta],x]]

In[4]:=
u1a=theta*(Tanh[(theta*x+G[y]-theta^2*t-Log[y])/2]+1)

In[5]:=
eta1a=(theta/2/y)*(y*D[G[y],y]-1)*Sech[(\
theta*x+G[y]-theta^2*t-Log[y])/2]^2-1

In[6]:=
Eqn1Verify[u1a,eta1a]

Out[6]=
0

In[7]:=
www[u1a,eta1a]

In[8]:=
Eqn2V[u1a,eta1a]

Out[8]=
0
```

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